J Math Chem (2012) 50:289–299
DOI 10.1007/s10910-011-9915-3

ORIGINAL PAPER

Theory and algorithm of the inversion method for pentadiagonal matrices

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Received: 29 June 2011 / Accepted: 8 September 2011 / Published online: 22 September 2011

Abstract A recently developed inversion method for pentadiagonal matrices is reconsidered in this work. The mathematical structure of the previously suggested method is fully developed. In the process of establishing the mathematical structure, certain determinantial relations specific to pentadiagonal matrices were also established. This led to a rather general necessary and sufficient condition for the method to work. All the so called forward and backward leading principal submatrices need to be non-singular. While this condition sounds restrictive it really is not so. These are in fact the conditions for forward and backward Gauss Eliminations without any pivoting requirement. Additionally, the method is more effective computational complexity wise then recently published competitive methods.

Keywords Direct methods for linear systems and matrix inversion · Difference equations · Matrices, determinants

1 Introduction

Tridiagonal matrices facilitate their inversion and eigenvalue problems due to their three-term-recursive natures. This feature may enable us to find general analytic solutions to these recursions. Nevertheless there exist a lot of numerical methods for these equations. In nonunderestimatedly many cases, five or four term recursions can be

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rather easily constructed and await being solved. Mathematical chemisty is one of the fields where such recursion may appear. Hence their solution, or in other words, the inversion of the pentadiagonal matrices gain importance even in mathematical chemistry [1]. This work is devoted to this task.

Here, we reconsider the inversion method for pentadiagonal matrices previously suggested by the present authors [2]. We give a necessary and sufficient condition for the method to work, and also discuss its computational cost. The results are applicable to Huang and McColl's case for tridiagonal matrices too [3]. Therefore, the range of applicability of their method is extended. First, the method is reintroduced briefly, and then the lemmas and their mathematical explanations are made. Finally, the necessary and sufficient condition is proven and the whole formulation for the method is shown as mathematical structures. In the last part, computational cost is analyzed.

2 The method

In this work we revisit the method developed by the present authors to construct the inverse of an $N \times N$ adjacent pentadiagonal matrix with real elements. The matrix under consideration is as follows.

$$A_N = \begin{bmatrix} c_1 & d_1 & e_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ b_2 & c_2 & d_2 & e_2 & 0 & \cdots & \cdots & \cdots & 0 \\ a_3 & \mathrm{b}_3 & c_3 & d_3 & e_3 & 0 & \cdots & \cdots & 0 \\ 0 & & & \ddots & & & \ddots & & \vdots \\ \vdots & 0 & & & \ddots & & & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & & & 0 \\ \vdots & & & \ddots & & & \ddots & & e_{N-2} \\ \vdots & & & & \ddots & & & \ddots & d_{N-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_N & b_N & c_N \end{bmatrix} \tag{1}$$

Although the method was given elsewhere [2] we shall briefly summarize it here. Define $a_1, a_2, b_1, d_N, e_{N-1}$ and e_N as zero for the sake of notation consistency. Let the inverse matrix A_N^{-1} be called Φ_N. To develop an algorithm consider the multiplication of the matrix A_N with the jth column of Φ_N. The product will correspond to the jth column of the N by$N$ unit matrix I_N. Using the notation and with the help of the recursive relationship in the multiplication $A_N \Phi_N$, the following second-order difference equations can be obtained.

$$\phi_N(i,j) = \begin{cases} -\left[X_{i+2}\phi_N(i+1,j) + \mathrm{Y}_{\mathrm{i+2}}\phi_N(i+2,j)\right], & \mathrm{i} < \mathrm{j} \\ -\left[\mathrm{T}_{col-i}\phi_N(i-1,j) + \mathrm{Z}_{\mathrm{col}-i}\phi_N(i-2,j)\right]. & \mathrm{i} > \mathrm{j} \end{cases} \tag{2}$$

where $X_1 \equiv 0, X_2 \equiv 0, X_3 \equiv d_1/c_1$ and $Y_1 \equiv 0, Y_2 \equiv 0, Y_3 \equiv e_1/c_1$ and also $col \equiv N+3, T_1 \equiv 0, T_2 \equiv 0, T_3 \equiv b_{col-2}/c_{col-3}$ and $Z_1 \equiv 0, Z_2 \equiv 0, Z_3 \equiv a_{col-2}/c_{col-3}$.

$$X_i = (d_{i-2} + X_{i-2}Y_{i-1}a_{i-2} - Y_{i-1}b_{i-2})/\alpha_i \tag{3}$$
$$Y_i = e_{i-2}/\alpha_i \tag{4}$$
$$\alpha_i = c_{i-2} + X_{i-2}X_{i-1}a_{i-2} - (X_{i-1}b_{i-2} + Y_{i-2}a_{i-2}) \tag{5}$$
$$T_i = (b_{col-i} + T_{i-2}Z_{i-1}e_{col-i} - Z_{i-1}d_{col-i})/\eta_i \tag{6}$$
$$Z_i = a_{col-i}/\eta_i \tag{7}$$
$$\eta_i = c_{col-i} + T_{i-2}T_{i-1}e_{i-2} - (T_{i-1}d_{col-i} + Z_{i-2}a_{col-i}) \tag{8}$$

For the sake of convenience $\alpha_1 \equiv 0, \alpha_2 \equiv 0, \eta_1 \equiv 0$ and $\eta_2 \equiv 0$ choices are made. α_i's are scaling factors required by the method and so are η_i's. (3), (4) and (5) are used for obtaining upper triangular elements of the inverse matrix, similarly (6), (7) and (8) are used for obtaining lower triangular elements of the inverse matrix. Solving these equations the following equalities are obtained.

$$\phi_N(i,i) = \left[\alpha_{i+2} - e_j Z_{N+1-i} - \frac{(X_{i+2}\alpha_{i+2} - e_i T_{N+1-i})(T_{N+2-i} - Z_{N+2-i}X_{i+1})}{1 - Z_{N+2-i}Y_{i+1}}\right]^{-1} \tag{9}$$

$$\phi_N(i+1,i) = \left[X_{i+2}\alpha_{i+2} - e_i T_{N+1-i} - \frac{(\alpha_{i+2} - e_i Z_{N+1-i})(1 - Z_{N+2-i}Y_{i+1})}{T_{N+2-i} - Z_{N+2-i}X_{i+1}}\right]^{-1} \tag{10}$$

Now, to start considering how these equations and relations can be used to suggest a method for calculating Φ_N, calculate all of the X_i, Y_i, T_i and Z_i values at the first step. At the next step, calculate the diagonal and subdiagonal elements of Φ_N, that is $\phi_N(i,i)$ and $\phi_N(i+1,i)$. Now start utilizing (2) to build up the upper elements of the inverse matrix Φ_N starting from the last element $\phi_N(N,N)$ and a hypothetical element $\phi_N(N+1,N)$ which is set equal to zero. When this is done, construct the elements in the upper part of Φ_N column by column starting from the element $\phi_N(N-2,N-1)$ upwards and so on. The same procedure follows for the lower part of Φ_N.

3 Conditions for the method to work

We shall start by defining forward and backward leading principal submatrices of an $N \times N$ pentadiagonal matrix A_N. In fact the forward leading principal submatrix A_i is the well-known ith leading principal submatrix obtained by deleting all rows and columns of A_N after its ith row and ith column. Similarly, the backward leading principal submatrix ${}_{(N-i)}A$ is obtained by deleting all rows and all columns of A_N up to its $(N-i)$th row and $(N-i)$th column.

Investigating the equalities which help us find the values of Φ_N, we realize that these equalities can have singularity problems since their denominators can become zero. The following lemmas lead to a theorem giving us the necessary and sufficient condition for the method to work.

Lemma 3.1 *For $i \geq 6$ $\Gamma_i, \mathrm{B}_i, \gamma_i$ and β_i are defined as follows.*

$$\Gamma_i = \mathrm{B}_{i-1}d_{i-2} - \mathrm{B}_{i-2}e_{i-3}b_{i-2} + \Gamma_{i-2}e_{i-3}a_{i-2} \tag{11}$$
$$\mathrm{B}_i = \mathrm{B}_{i-1}c_{i-2} - \Gamma_{i-1}b_{i-2} + a_{i-2}\left[\Gamma_{i-2}d_{i-3} - e_{i-4}(\mathrm{B}_{i-3}e_{i-3} - \mathrm{B}_{i-4}a_{i-3}e_{i-5})\right] \tag{12}$$
$$\gamma_i = \beta_{i-1}b_{col-i} - \beta_{i-2}a_{col-i+1}d_{col-i} + \gamma_{i-2}a_{col-i+2}e_{col-i} \tag{13}$$

$$\beta_i = \beta_{i-1}c_{col-i} - \gamma_{i-1}d_{col-i} + e_{col-i}$$
$$\left[\gamma_{i-2}b_{col-i+1} - a_{col-i+2}\left(\beta_{i-3}c_{col-i+1} - \beta_{i-4}a_{col-i+3}e_{col-i+1}\right)\right] \tag{14}$$

Assume G_{i-2} is the matrix obtained after deleting $(i-1)$th row and $(i-2)$th column of the pentadiagonal matrix A_{i-1}. Γ_i and B_i are equal to the determinants of the G_{i-2} and A_{i-2}, respectively. Similarly, ${}_1A$ is the $(N-1)\times(N-1)$ backward leading principal pentadiagonal submatrix. ${}_2H$ is the $(N-2)\times(N-2)$ matrix obtained by deleting the $(N+2-i)$th row and the $(N+3-i)$th column of ${}_1A$. γ_i and β_i correspond to the determinans of ${}_2H$ and ${}_2A$ respectively.

Proof Special equalities can be written for the case $i < 6$. We shall use induction to prove the lemma by assuming the equalities to be valid for all the steps up to $(i-1)$ and show that they are true for the ith step.

Assume that the theorem is valid up to Γ_{i-1} and B_{i-1}. Consider the $(i-1)$ dimensional pentadiagonal matrix and remove the $(i-1)$th row and the $(i-2)$th column. This can be shown as

$$A_{i-1}(i-1,i-2) = \begin{bmatrix} c_1 & & & & & & 0 & 0 \\ & \ddots & & & & & \vdots & \vdots \\ & & \ddots & & & & \vdots & \vdots \\ & & & \ddots & & & 0 & 0 \\ & & & & \ddots & & e_{i-5} & 0 \\ & & & & & c_{i-4} & d_{i-4} & 0 \\ \hline 0 & \cdots & \cdots & 0 & a_{i-3} & b_{i-3} & c_{i-3} & e_{i-3} \\ \hline 0 & \cdots & \cdots & 0 & 0 & a_{i-2} & b_{i-2} & d_{i-2} \\ \hline & & & & & & & \end{bmatrix}_{(i-2)\times(i-2)} \tag{15}$$

Starting the evaluation of the determinant from d_{i-2}, we begin with $\mathrm{B}_{i-1}d_{i-2}x$. If we continue on the classical determinant process with e_{i-3}, the result is $\mathrm{B}_{i-2}b_{i-2}e_{i-3}$. The last additive expression is $\Gamma_{i-2}e_{i-3}a_{i-2}$ for a_{i-2}. Summations of these expressions give the equality (11).

For proving equality (12), we concentrate on the determinant of the $(i-2)$ dimensional adjacent pentadiagonal matrix. Starting with c_{i-2} gives the expression $\mathrm{B}_{i-1}c_{i-2}$. Similarly continuing with b_{i-2} results in the expression $\Gamma_{i-1}b_{i-2}$. The last part of the determinant is the part with the multiplier a_{i-2}.

$$A_{i-2} = \left[\begin{array}{ccccc|c|cc} c_1 & & & & & 0 & 0 & 0 \\ & \ddots & & & & \vdots & \vdots & \vdots \\ & & \ddots & & & 0 & 0 & 0 \\ & & & \ddots & & e_{i-6} & 0 & 0 \\ & & & & c_{i-5} & d_{i-5} & e_{i-5} & 0 \\ 0 & \cdots & 0 & a_{i-4} & b_{i-4} & c_{i-4} & d_{i-4} & e_{i-4} \\ 0 & \cdots & \cdots & 0 & a_{i-3} & b_{i-3} & c_{i-3} & d_{i-3} \\ \hline 0 & \cdots & \cdots & \cdots & 0 & a_{i-2} & b_{i-2} & c_{i-2} \end{array}\right] \tag{16}$$

Using the matrix in (16), the last part for the determinant is $\mathrm{B}_{i-4}a_{i-2}a_{i-3}e_{i-4}e_{i-5}$. All these, give the equality (12).

A similar treatment gives equalities (13) and (14).

Lemma 3.2 *Equalities (3), (4), (6) and (7) in the method section are respectively expressed as follows.*

$$X_i = \Gamma_i/\mathrm{B}_i \tag{17}$$

$$Y_i = \left(\mathrm{B}_{i-1}e_{i-2}\right)/\mathrm{B}_i \tag{18}$$

These yield

$$\alpha_i = \mathrm{B}_i/\mathrm{B}_{i-1} \tag{19}$$

Proof If we start with the case $i < j$ for $1 \leq i \leq k-1 < j-1$, equality (2) gives

$$\phi_N(k-1, j) = -\left[X_{k+1}\phi_N(k, j) + Y_{k+1}\phi_N(k+1, j)\right] \tag{20}$$

$$\phi_N(k-2, j) = -\left[X_k\phi_N(k-1, j) + Y_k\phi_N(k, j)\right] \tag{21}$$

for $i = k-1$ and $i = k-2$, respectively. For the proof we assume that it is true up to the $(k-1)$th step, and show that it is true for the kth step. The result of multiplying the kth row of A_N with the kth column of Φ_N is

$$\begin{aligned} &a_k\phi_N(k-2, j) + b_k\phi_N(k-1, j) + c_k\phi_N(k, j) + d_k\phi_N(k+1, j) \\ &\quad + e_k\phi_N(k+2, j) = 0 \end{aligned} \tag{22}$$

Using (20) and (21) in (22) gives

$$\begin{aligned} &\left[c_k + X_kX_{k+1}a_k - (X_{k+1}b_k + Y_ka_k)\right]\phi_N(k, j) \\ &+ (d_k + X_kY_{k+1}a_k - Y_{k+1}b_k)\,\phi_N(k+1, j) + e_k\phi_N(k+2, j) = 0 \end{aligned} \tag{23}$$

After multiplication of B_{k+1} with Eq. (23) the following equation is obtained.

$$\begin{aligned} &\left[\mathrm{B}_{k+1}c_k + \mathrm{B}_{k+1}X_kX_{k+1}a_k - \mathrm{B}_{k+1}\left(X_{k+1}b_k + Y_ka_k\right)\right]\phi_N(k, j) + \\ &\quad + \left(\mathrm{B}_{k+1}d_k + \mathrm{B}_{k+1}X_kY_{k+1}a_k - \mathrm{B}_{k+1}Y_{k+1}b_k\right)\phi_N(k+1, j) \\ &\quad + \mathrm{B}_{k+1}e_k\phi_N(k+2, j) = 0 \end{aligned} \tag{24}$$

For proving (17) and (18), the equation above must be equal to the following equation obtained by (2) for the k^{th} step.

$$\phi_N(k, j) = -\left[X_{k+2}\phi_N(k+1, j) + Y_{k+2}\phi_N(k+2, j)\right] \tag{25}$$

The above equality can be rewritten with the help of (17) and (18).

$$\mathrm{B}_{k+2}\phi_N(k,j)+\Gamma_{k+2}\phi_N(k+1,j)+\mathrm{B}_{k+2}e_{k-2}\phi_N(k+2,j)=0 \tag{26}$$

If (24) and (26) are equal, the coefficients must be equal. Firstly, we start with the coefficients of $\phi_N(k+1,j)$ in (24) and (26). It can be easily seen that these coefficients are equal since from the induction hypothesis, we already assumed that $X_k=\Gamma_k/\mathrm{B}_k$ and $Y_{k+1}=(\mathrm{B}_k e_{k-1})/\mathrm{B}_{k+1}$ are true.

If the coefficients of $\phi_N(k,j)$ in (24) and (26) are equal, proof will be complete for (17), (18) and (19). We assume that it is true up to the $(k-1)$th step. Similarly, multiplying the $(k-1)$th row of A_N with the $(k-1)$th column of Φ_N gives the equality.

$$\begin{aligned}&a_{k-1}\phi_N(k-3,j)+b_{k-1}\phi_N(k-2,j)+c_{k-1}\phi_N(k-1,j)+d_{k-1}\phi_N(k,j)\\&\quad+e_{k-1}\phi_N(k+1,j)=0\end{aligned} \tag{27}$$

Using (2) into (30) gives

$$\begin{aligned}&\left[c_{k-1}+X_{k-1}X_k a_{k-1}-(X_k b_{k-1}+Y_{k-1}a_{k-1})\right]\phi_N(k-1,j)\\&+(d_{k-1}+X_{k-1}Y_k a_{k-1}-Y_k b_{k-1})\,\phi_N(k,j)+e_{k-1}\phi_N(k+1,j)=0\end{aligned} \tag{28}$$

After multiplying with B_k, the equation becomes

$$\begin{aligned}&\left[\mathrm{B}_k c_{k-1}+\mathrm{B}_k X_{k-1}X_k a_{k-1}-\mathrm{B}_k\,(X_k b_{k-1}+Y_{k-1}a_{k-1})\right]\phi_N(k-1,j)\\&+(\mathrm{B}_k d_{k-1}+\mathrm{B}_k X_{k-1}Y_k a_{k-1}-\mathrm{B}_k Y_k b_{k-1})\,\phi_N(k,j)+\mathrm{B}_k e_{k-1}\phi_N(k+1,j)=0\end{aligned} \tag{29}$$

Using $X_{k-1}=\Gamma_{k-1}/\mathrm{B}_{k-1}$ and $Y_{k-1}=(\mathrm{B}_{k-2}e_{k-3})/\mathrm{B}_{k-1}$ in (2), the equation

$$\phi_N(k-1,j)=-\left[(\Gamma_{k+1}/\mathrm{B}_{k+1})\,\phi_N(k,j)+\left[(\mathrm{B}_k e_{k-1})/\mathrm{B}_{k+1}\right]\phi_N(k+1,j)\right] \tag{30}$$

is obtain. This equation can be rewritten as follows.

$$\mathrm{B}_{k+1}\phi_N(k-1,j)+\Gamma_{k+1}\phi(k,j)+\mathrm{B}_k e_{k-1}\phi_N(k+1,j)=0 \tag{31}$$

The coefficient of (29) and (31) must be equal. Then we investigate the coefficients of $\phi_N(k+1,j)$. If the coefficients are equal, we can write the following equality with the help of the formulations for X_{k-1}, X_k and Y_{k-1} formulations.

$$\begin{aligned}&\mathrm{B}_{k-1}\mathrm{B}_k c_{k-1}-\Gamma_{k-1}\Gamma_k a_{k-1}-\mathrm{B}_{k-1}\Gamma_k b_{k-1}-\mathrm{B}_{k-2}\mathrm{B}_k a_{k-1}e_{k-3}\\&\quad=\mathrm{B}_{k-1}\mathrm{B}_k c_{k-1}-\mathrm{B}_{k-1}\Gamma_k b_{k-1}+\mathrm{B}_{k-1}a_{k-1}\\&\qquad\times\left[\Gamma_{k-1}d_{k-2}-e_{k-3}\,(\mathrm{B}_{k-2}e_{k-2}-\mathrm{B}_{k-3}a_{k-2}e_{k-4})\right]\end{aligned} \tag{32}$$

This equality can be rewriten as

$$\mathrm{B}_{k-1}\mathrm{B}_k c_{k-1} - \mathrm{B}_{k-2}\mathrm{B}_k a_{k-1} e_{k-3} = \mathrm{B}_{k-1}\mathrm{B}_{k+1} + \Gamma_{k-1}\Gamma_k a_{k-1} + \mathrm{B}_{k-1}\Gamma_k b_{k-1} \tag{33}$$

After multiplying with X_k and adding $\mathrm{B}_k \Gamma_k d_{k-1}$, the above equality turns into

$$\begin{aligned} &\mathrm{B}_k \Gamma_k d_{k-1} - \mathrm{B}_k e_{k-2} \left(\mathrm{B}_{k-1} c_{k-1} - \mathrm{B}_{k-2} e_{k-3} a_{k-1}\right) \\ &= \Gamma_k \left(\mathrm{B}_k d_{k-1} + \Gamma_{k-1} a_{k-1} e_{k-2} - \mathrm{B}_{k-1} b_{k-1} e_{k-2}\right) - \mathrm{B}_{k-1}\mathrm{B}_{k+1} e_{k-2} \end{aligned} \tag{34}$$

On the right hand side of (34), the term in the paranthesis equals Γ_{k+1}. After multiplying with a_k/B_k and adding $\mathrm{B}_{k+1} c_k - \Gamma_{k+1} b_k$, the following equality is obtained.

$$\begin{aligned} &\mathrm{B}_{k+1} c_k - \Gamma_{k+1} b_k + a_k \left[\Gamma_k d_{k-1} - e_{k-2} \left(\mathrm{B}_{k-1} c_{k-1} - \mathrm{B}_{k-2} e_{k-3} a_{k-1}\right)\right] \\ &= \mathrm{B}_{k+1} c_k - \Gamma_{k+1} b_k + \left(\Gamma_k \Gamma_{k+1}/\mathrm{B}_k\right) a_k - \left(\mathrm{B}_{k-1}\mathrm{B}_{k+1} e_{k-2}/\mathrm{B}_k\right) a_k \end{aligned} \tag{35}$$

Multiplying the last two terms at the right hand side of (35) by $\mathrm{B}_{k+1}/\mathrm{B}_{k+1}$ will not alter the equality.

$$\begin{aligned} &\mathrm{B}_{k+1} c_k - \Gamma_{k+1} b_k + a_k \left[\Gamma_k d_{k-1} - e_{k-2} \left(\mathrm{B}_{k-1} e_{k-1} - \mathrm{B}_{k-2} a_{k-1} e_{k-3}\right)\right] \\ &\quad = \mathrm{B}_{k+1} c_k - \left(\Gamma_{k+1}/\mathrm{B}_{k+1}\right) \mathrm{B}_{k+1} b_k + \left(\Gamma_{\mathrm{k+1}} \Gamma_{\mathrm{k}}/\mathrm{B}_{\mathrm{k}} + 1\mathrm{B}_{\mathrm{k}}\right) \mathrm{B}_{\mathrm{k}} + 1\mathrm{a}_{\mathrm{k}} \\ &\quad\quad - \left(\mathrm{B}_{k-1} e_{k-2}/\mathrm{B}_k\right) \mathrm{B}_{k+1} a_k \end{aligned} \tag{36}$$

The left hand side of the equality equals B_{k+2} and the right hand side equals the coefficients of $\phi_N(k, j)$ in (24) with the help of X_k, X_{k+1} and Y_k.

Similarly, $T_i = \gamma_i/\beta_i$, $Z_i = \left(\beta_{i-1} a_{col-i}\right)/\beta_i$ and $\eta_i = \beta_i/\beta_{i-1}$ can be proven.

Theorem 3.1 *The equalities used to find the diagonal and the first subdiagonal elements of the inverse matrix* Φ_N *can be written as*

$$\phi_N(i, i) = \Lambda_i/(\Xi_i \Lambda_i - \Psi_i \Omega_i) \tag{37}$$

$$\phi_N(i+1, i) = \Omega_i/(\Psi_i \Omega_i - \Xi_i \Lambda_i) \tag{38}$$

where $\Lambda_i = 1 - Z_{N+2-i} Y_{i+1}$, $\Omega_i = T_{N+2-i} - Z_{N+2-i} X_{i+1}$, $\Psi_i = X_{i+2}\alpha_{i+2} - e_i T_{N+1-i}$ *and* $\Xi_i = \alpha_{i+2} - e_i Z_{N+1-i}$.

(37) and (38) are not singular unless the determinants of the forward and backward leading principal submatricies of A_N *are zero.*

Proof First, we concentrate on Λ_i and Ω_i, the numerators of (37) and (38). The expressions can be re-written as follows using expansions of Z_{N+2-i}, T_{N+2-i}, Y_{i+1} and X_{i+1}.

$$\Lambda_i = \left(\beta_{N+2-i}\mathrm{B}_{i+1} - \beta_{N+1-i}\mathrm{B}_i a_{i+1} e_{i-1}\right) / \left(\beta_{N+2-i}\mathrm{B}_{i+1}\right) \tag{39}$$

$$\Omega_i = \left(\gamma_{N+2-i}\mathrm{B}_{i+1} - \beta_{N+1-i}\Gamma_{i+1}\right)/\left(\beta_{N+2-i}\mathrm{B}_{i+1}\right) \tag{40}$$

The denominators of (39) and (40) are multiplications of determinants of the forward and backward leading principal submatrices of A_N. If these determinants are not equal to zero, (39) and (40) are not singular.

The proof will be complete after showing that $\Xi_i \Lambda_i - \Psi_i \Omega_i$ is not equal to zero if the determinants are not equal to zero. For the proof, the following lemma is neccessary.

Lemma 3.3

$$\Xi_i \Lambda_i - \Psi_i \Omega_i = \mathrm{B}_{N+2}/(\mathrm{B}_{N+1}\beta_{N+2-i}) \tag{41}$$

Proof The equalities Λ_i, Ω_i, Ψ_i and Ξ_i can be reorganized as

$$\Lambda_i = \lambda_i/\beta_{N+2-i}\mathrm{B}_{i+1} \tag{42}$$
$$\Omega_i = \omega_i/\beta_{N+2-i}\mathrm{B}_{i+1} \tag{43}$$
$$\Psi_i = \psi_i/\beta_{N+1-i}\mathrm{B}_{i+1} \tag{44}$$
$$\Xi_i = \xi_i/\beta_{N+1-i}\mathrm{B}_{i+1} \tag{45}$$

$\lambda_i = \beta_{N+2-i}\mathrm{B}_{i+1} - \beta_{N+1-i}\mathrm{B}_i a_{i+1} e_{i-1}$, $\omega_i = \gamma_{N+2-i}\mathrm{B}_{i+1} - \beta_{N+1-i}\Gamma_{i+1}$, $\psi_i = \Gamma_{i+2}\beta_{N+1-i} - \gamma_{N+1-i}\mathrm{B}_{i+1}e_i$ and $\xi_i = \beta_{N+1-i}\mathrm{B}_{i+2} - \beta_{N-i}\mathrm{B}_{i+1}e_i$ in here. With the help of these expressions, equality (41) can be re-written as follows

$$(\xi_i\lambda_i - \Psi_i\omega_i)/(\beta_{N+1-i}\mathrm{B}_{i+1}) = \mathrm{B}_{N+2} \tag{46}$$

With the help of Laplace Expansion, ξ_i is equal to the determinant of K_{N-1} which is the matrix obtained after deleting $(i+1)$th row and $(i+1)$th column of the pentadiagonal matrix A_N. Similarly λ_i, ω_i and ψ_i are equal to the determinants of the matrix obtained respectively by deleting the ith row and the ith column of A_N, the ith row and the $(i+1)$th column of A_N and $(i+1)$th row and ith column of A_N. With the help of the Sylvester Determinant Identity, equality (46) is valid. Since B_{N+2} is one of the leading principal submatrices of A_N, the left hand side of (41) are not singular unless the determinants of principal submatrices are zero.

Although the necessary and sufficient condition given by the theorem seems to be somewhat restrictive, it really is not so. The former of these conditions, that is the forward leading principal submatrices being non-singular is the requirement for Gauss elimination by creating zeroes under the main diagonal without any need for pivoting. The latter of these conditions, that is the backward leading principal submatrices being non-singular is the requirement for Gauss elimination by creating zeroes above main diagonal without any need for pivoting. It is quite evident that need for pivoting will result in the band structure of the original matrix, in this case a pentadiagonal envelope.

4 Performance analysis of the method

In this section, computational cost is shown with the help of algorithmic and theorical explanations. To explain the algorithm, a pseudo code of the method is given below.

```
Input arrays a, b, c, d and e of the matrix A_N
Assign X_1 = 0, X_2 = 0, Y_1 = 0, Y_2 = 0, T_1 = 0, T_2 = 0, Z_1 = 0,
Z_2 = 0, α_1 = 0, α_2 = 0, η_1 = 0 and η_2 = 0.

For i = 2 to N
  Calculate X_i, Y_i, T_i, Z_i, α_i and η_i
EndFor

For i = 1 toN − 1
  Calculate Λ_i, Ω_i, Ψ_i and Ξ_i
  Calculate φ_N(i, i)
  Calculate φ_N(i + 1, i)
EndFor

For i = 3 toN
  For j = 1 to i − 2
    Calculate φ_N(i, j)
  EndFor
EndFor

For i = N + 3 to 1
  For j = i + 1 to N
    Calculate φ_N(i, j)
  EndFor
EndFor
```

The computational cost of the first "for" cycle is $33(N - 1)$. According to Lemma 3.2, it looks as if some of the fractions used are unnecessary, however this may not be true in certain cases. It has been observed during numerical applications that the recursive relations used during the construction of the inverse of pentadiagonal matrix may result in numerical values to grow in an uncontrolled manner. To avoid this, scaling factors were used. Since these scalings bring only an additional expense of $O(N)$, they are of tolerable nature.

Cost of the second "for" cycle arises from obtaining diagonal and first subdiagonal elements of the inverse matrix Φ_N which is $20(N - 1)$. With the help of these computations, to find all elements of the inverse matrix Φ_N the computational cost is only $3N^2 + 2N - 5$. Together with certain additional minor calculations the total cost turns out to be $3N^2 + 54N - 36$.

There are two reasonably recent publications on this subject. The first of these is by Zhao and Huang [4]. The computational complexity of their method is $O(N^3)$ which is quite costly in comparison to the $O(N^2)$ method suggested by the present authors [2]. It is no surprise that even for relatively small N values the method of reference [4] is incomparibly slower than that of reference [2] which was published by the present authors half a decade prior to reference. The result of reference [4] are compared in Fig. 1 with those of the one discussed here.

The second of these methods is that of Hadj and Elonafi [6]. It is clearly much faster than that of reference [4]. However it is not clear how the authors claim that their method is of order $O(N)$. Our implementations show an $O(N^2)$ behaviour for their method and which is that of the order of the method discussed in the present article also. A comparison of the method of reference [6] and that of the present method

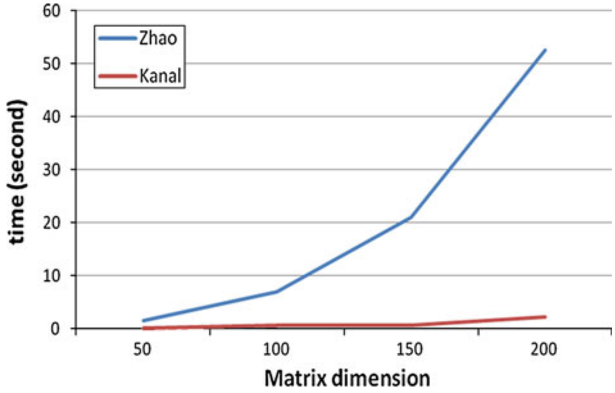

Fig. 1 Time comparison of Zhao and Huang method [4] with the present method

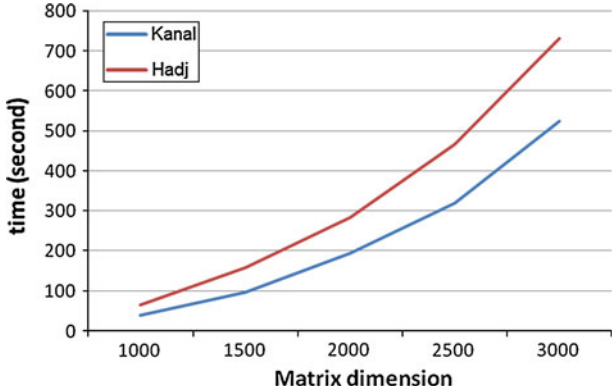

Fig. 2 Time comparison of Hadj and Elounafi method [6] with the present method

is given in Fig. 2. The N^2 behaviour is quite evident with the parabolic shape of both methods. However the two graphs favour the present method.

The pseudo code can be organized according to the different structures of C and Fortran programming languages. In the parallel code of the method [4], elements of the inverse matrix was constructed row by row in accordance with the structure of C language.

5 Conclusion

The mathematical structure of an inversion method previously suggested by the authors [2] and recently adopted parallel processing [4] by one of the authors is further analyzed here, leading to the rather general necessary and sufficient condition for the method to work.

An important factor which makes the method superior to most other methods is due to its low computational cost. The calculation of the inverse of the pentadiagonal matrices requires the solution of second order difference equations. The method keeps the main computational cost at $O(N^2)$ with a coefficient value 3. This results in a total computational cost of $3N^2 + 54N - 36$ and is the cheapest of its kind [5,6].

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